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ON THE RELATIONSHIP BETWEEN THE LOGARITHMIC STRAIN RATE AND THE STRETCHING TENSOR

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Abstract—In this paper we investigate the relationship between the stretching tensor **D** and the logarithmic (Hencky) strain ln **V**, with **V** the left stretch tensor. We establish the simple formula

$$\mathbf{D} = (\ln \mathbf{V})^{\circ} - \operatorname{sym}(\mathbf{F}\boldsymbol{\Omega}_r\mathbf{F}^{-1}),$$

which holds for arbitrary three-dimensional motions. Here **F** is the deformation gradient, $(\ln \mathbf{V})^{\circ}$ is the time derivative of ln **V** measured in a coordinate system which rotates with the left principal strain axes, and $\boldsymbol{\Omega}_r$ is the spin of the right principal strain axes. We use this formula to show that $\mathbf{D} = (\ln \mathbf{V})^{\circ}$, (or, equivalently, $\mathbf{D} = (\ln \mathbf{V})^{*}$, the Jaumann derivative of ln **V**), if and only if the characteristic spaces of the right stretch tensor **U** are constant on any time interval in which the number of distinct principal stretches is constant. Finally, we show that the asymptotic approximation

$$\mathbf{D} = (\ln \mathbf{V})^{*} + O(\epsilon^3)$$

holds whenever the displacement gradient **H** satisfies $\mathbf{H} = O(\epsilon)$, $\dot{\mathbf{H}} = O(\epsilon)$.

1. INTRODUCTION

The logarithmic strain introduced by Hencky[3] has long enjoyed favored treatment in the metallurgical and materials science literature, where it is referred to as the "true" or "natural" strain. Its use, however, has been primarily limited to studies in which the principal axes of strain are fixed. In such problems, the simple relationship

$$\mathbf{D} = (\ln \mathbf{V})^{\cdot}$$

exists between the stretching tensor **D** and the logarithm of the left stretch tensor **V**.

In this paper, we investigate the question of whether an analogous relationship exists for general three-dimensional deformations. Truesdell and Toupin ([9], pp. 269–270) note that the Hencky strain has never been successfully applied in general. Recent attempts to remedy this situation include the work of Hill[4], who argues that logarithmic strain measures have inherent advantages in certain constitutive inequalities. He finds the rather complicated relation

$$(\ln U)_{ij}^{\cdot} = \begin{cases} D_{ii}, & i = j, \\ 2(\lambda_i\lambda_j \ln(\lambda_i/\lambda_j) D_{ij})/(\lambda_i^2 - \lambda_j^2), & i \neq j, \end{cases}$$

(no summation) with the components of $(\ln \mathbf{U})^{\cdot}$ and **D** taken with respect to eigenvector bases of **U** and **V**, respectively. (Here **U** is the right stretch tensor and the λ_i are the principal stretches; i.e. the eigenvalues of **U**.)

Both Hutchinson and Neale[5] and Stören and Rice[8] find Hencky strain to be useful in the formulation of the deformation theory of plasticity, although Stören and Rice decide that the general relation between D and $(\ln \mathbf{U})^{\cdot}$ is "very complicated". They conclude that ln **U** is "essentially intractable" as a strain measure. Fitzgerald[1] decides that the utility of logarithmic strain is limited to problems with fixed principal strain axes. In addition to this negative concensus on the applicability of Hencky strain measures, the above authors all consider only strains for which the principal stretches are distinct, and do not rigorously treat the possibility that the principal axes of strain may not be uniquely defined.

We here attempt to give a complete answer to the general three-dimensional problem. We begin by establishing a simple general formula relating $\mathbf{D}$ and $\ln \mathbf{V}$:

$$\mathbf{D} = (\ln \mathbf{V})^{\circ} - \operatorname{sym}(\mathbf{F}\boldsymbol{\Omega}_r\mathbf{F}^{-1}).$$

Here $\mathbf{F}$ is the deformation gradient, $\boldsymbol{\Omega}_r$ is the spin of the right principal axes of strain, and $(\ln \mathbf{V})^{\circ}$ is the time derivative of $\ln \mathbf{V}$ measured by an observer rotating with the left principal axes of strain. We use this formula to prove that $\mathbf{D} = (\ln \mathbf{V})^{\circ}$ (or, equivalently, $\mathbf{D} = (\ln \mathbf{V})^{*}$, the Jaumann derivative of $\ln \mathbf{V}$), if and only if the characteristic spaces of $\mathbf{U}$ are independent of time on any time interval in which the number of distinct principal stretches is constant. Finally, we show that for motions which are small in the sense that the displacement gradient $\mathbf{H} = \mathbf{F} - \mathbf{I}$ satisfies $\mathbf{H} = O(\epsilon)$, $\dot{\mathbf{H}} = O(\epsilon)$, we have the estimate

$$\mathbf{D} = (\ln \mathbf{V})^{*} + O(\epsilon^3).$$

Since $\mathbf{D} = O(\epsilon)$ and $(\ln \mathbf{V})^{*} = O(\epsilon)$, we conclude that $(\ln \mathbf{V})^{*}$ is, in fact, an excellent approximation to $\mathbf{D}$ in this instance.

2. PRELIMINARIES†

Consider a motion of a body, and let $\mathbf{F}(t)$ denote the *deformation gradient* corresponding to a given material point. At each t (in a fixed time interval) $\mathbf{F}(t)$ is a 3×3 matrix with strictly-positive determinant, and hence admits the polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

where $\mathbf{U}(t)$ and $\mathbf{V}(t)$ are symmetric, positive-definite, while $\mathbf{R}(t)$ is proper orthogonal. $\mathbf{U}$ and $\mathbf{V}$, respectively, are called the *right* and *left stretch tensors.* We assume that $\mathbf{F}$ is smooth (i.e. continuously differentiable); then

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{1}$$

is the *velocity gradient*,

$$\mathbf{D} = \operatorname{sym}\mathbf{L} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$$

the *stretching tensor*, and

$$\mathbf{W} = \operatorname{skw}\mathbf{L} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)$$

the *spin tensor.* (Here $\mathbf{L}^T$ denotes the transpose of $\mathbf{L}$.)

The *principal stretches* are the eigenvalues λ_i, $i = 1, 2, 3$, of $\mathbf{U}$ (or $\mathbf{V}$); since $\mathbf{U}$ is smooth (see Ref. 2, p. 23) we may, without loss in generality, choose the three functions $\lambda_i(t)$ to be smooth in t (see Kato[6], Thm. 6.8, p. 111). An orthonormal basis $\{\mathbf{r}_i\}$ of eigenvectors of $\mathbf{U}$ (where $\mathbf{r}_i$ corresponds to λ_i) is called a *right principal basis.* We assume that one such basis $\{\mathbf{r}_i\}$ is given, and that each $\mathbf{r}_i(t)$ is smooth‡ in t. Then $\{\mathbf{l}_i\}$ defined by

$$\mathbf{l}_i = \mathbf{R}\mathbf{r}_i$$

is a *left principal basis*; that is, $\{\mathbf{l}_i\}$ is an orthonormal basis of eigenvectors of $\mathbf{V}$ corresponding

†We follow the notation and terminology of [2].

‡This is an assumption: $\mathbf{U}$ smooth does not necessarily yield the existence of a smooth basis $\{\mathbf{r}_i\}$ (see Kato[6], Example 5.9, p. 115).

to $\{\lambda_i\}$. In view of the spectral theorem (see, e.g. Ref. 2, p. 11), we have the representations

$$\mathbf{U} = \lambda_i \mathbf{r}_i \otimes \mathbf{r}_i, \quad \mathbf{V} = \lambda_i \mathbf{l}_i \otimes \mathbf{l}_i.$$

Here $\mathbf{a} \otimes \mathbf{b}$ with components $a_i b_j$ is the tensor product of $\mathbf{a}$ and $\mathbf{b}$, and summation over repeated indices is implied. Also,

$$\mathbf{F} = \lambda_i \mathbf{l}_i \otimes \mathbf{r}_i. \tag{2}$$

The *characteristic space* for $\mathbf{U}$ at time t, corresponding to the principal stretch λ_i, say, is the set of *all* vectors $\mathbf{v}$ such that

$$\mathbf{U}(t)\mathbf{v} = \lambda_i(t)\mathbf{v},$$

so that coincident principal stretches correspond to the same characteristic space.

Let $\{\mathbf{e}_i\}$ be a smooth, time-dependent orthonormal basis. Then the corresponding *twirl tensor* $\mathbf{\Omega}$ is the skew tensor function defined by

$$\dot{\mathbf{e}}_i = \mathbf{\Omega}\mathbf{e}_i.$$

($\mathbf{\Omega}$ is skew since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ implies that $\Omega_{ij} + \Omega_{ji} = \mathbf{e}_i \cdot \dot{\mathbf{e}}_j + \dot{\mathbf{e}}_i \cdot \mathbf{e}_j = 0$.) Given any smooth tensor function $\mathbf{A}$, the *co-rotational derivative* $\mathbf{A}^\circ$ *of* $\mathbf{A}$ *relative to* $\{\mathbf{e}_i\}$ is defined by

$$\mathbf{A}^\circ = \dot{\mathbf{A}} + \mathbf{A}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{A}$$

and represents the time derivative of $\mathbf{A}$ measured by an observer rotating with $\{\mathbf{e}_i(t)\}$. Another important notion is the *Jaumann derivative* A^* of A, given by

$$\mathbf{A}^* = \dot{\mathbf{A}} + \mathbf{A}\mathbf{W} - \mathbf{W}\mathbf{A},$$

with $\mathbf{W}$ the spin (see, e.g. Ref. 7, p. 155).

Finally, the *tensor logarithm*, ln, maps symmetric, positive-definite matrices into symmetric matrices and is defined to be the inverse of the exponential function. In particular,

$$\ln \mathbf{U} = (\ln \lambda_i)\mathbf{r}_i \otimes \mathbf{r}_i, \qquad \ln \mathbf{V} = (\ln \lambda_i)\mathbf{l}_i \otimes \mathbf{l}_i. \tag{3}$$

3. RELATIONSHIP BETWEEN D AND (ln V)

Theorem 1. *Let* $\mathbf{\Omega}_r$ *denote the twirl tensor corresponding to* $\{\mathbf{r}_i\}$ *and* $(\ )^\circ$ *the co-rotational derivative relative to* $\{\mathbf{l}_i\}$. *Then*

$$\mathbf{D} = (\ln \mathbf{V})^\circ - \operatorname{sym}(\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}). \tag{4}$$

Proof. Differentiation of $(3)_2$ gives

$$\begin{aligned}(\ln \mathbf{V})^\cdot &= (\dot{\lambda}_i/\lambda_i)\mathbf{l}_i \otimes \mathbf{l}_i + (\ln \lambda_i)\dot{\mathbf{l}}_i \otimes \mathbf{l}_i + (\ln \lambda_i)\mathbf{l}_i \otimes \dot{\mathbf{l}}_i \\ &= (\dot{\lambda}_i/\lambda_i)\mathbf{l}_i \otimes \mathbf{l}_i + \mathbf{\Omega}_l(\ln \mathbf{V}) - (\ln \mathbf{V})\mathbf{\Omega}_l,\end{aligned}$$

with $\mathbf{\Omega}_l$ the twirl tensor corresponding to $\{\mathbf{l}_i\}$. Thus,

$$(\ln \mathbf{V})^\circ = (\dot{\lambda}_i/\lambda_i)\mathbf{l}_i \otimes \mathbf{l}_i.$$

Next, by (1) and (2),

$$\begin{aligned}\mathbf{L} &= (\dot{\lambda}_i\mathbf{l}_i \otimes \mathbf{r}_i + \lambda_i\dot{\mathbf{l}}_i \otimes \mathbf{r}_i + \lambda_i\mathbf{l}_i \otimes \dot{\mathbf{r}}_i)\left(\frac{1}{\lambda_j}\mathbf{r}_j \otimes \mathbf{l}_j\right) \\ &= (\dot{\lambda}_i/\lambda_i)\mathbf{l}_i \otimes \mathbf{l}_i + (\mathbf{\Omega}_l\mathbf{F} - \mathbf{F}\mathbf{\Omega}_r)\mathbf{F}^{-1} \qquad (5) \\ &= (\ln \mathbf{V})^\circ + \mathbf{\Omega}_l - \mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}.\end{aligned}$$

This completes the proof, as (4) is the symmetric part of (5).

Since

$$(\ln \mathbf{V})^* = (\ln \mathbf{V})^{\cdot} + (\ln \mathbf{V})\mathbf{W} - \mathbf{W}(\ln \mathbf{V}),$$

(4) may also be written in the form

$$\mathbf{D} = (\ln \mathbf{V})^* + (\ln \mathbf{V})(\mathbf{\Omega}_l - \mathbf{W}) - (\mathbf{\Omega}_l - \mathbf{W})(\ln \mathbf{V}) - \operatorname{sym}(\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}); \qquad (6)$$

we will use (6) in later calculations.

Remark. The term $\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}$ represents the spin of the right principal basis $\{\mathbf{r}_i\}$ as measured by an observer deforming with the body. (Let $\{\mathbf{v}_i\}$ be a basis fixed in space, and $\{\mathbf{b}_i\}$ a basis deforming with the body, i.e. $\mathbf{b}_i = \mathbf{F}\mathbf{v}_i$. Then the components of $\mathbf{\Omega}_r$ relative to $\{\mathbf{v}_i\}$ are the same as those of $\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}$ relative to $\{\mathbf{b}_i\}$.)

Remark. Similar arguments show that

$$\mathbf{D} = \mathbf{R}(\ln \mathbf{U})^\circ\mathbf{R}^T - \operatorname{sym}(\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}),$$

where $(\ )^\circ$ is the co-rotational derivative relative to $\{\mathbf{r}_i\}$, rather than $\{\mathbf{l}_i\}$.

4. WHEN DOES $\mathbf{D} = (\ln \mathbf{V})^\circ$?

In this section we present a condition on the principal strain axes under which the formula $\mathbf{D} = (\ln \mathbf{V})^\circ$ is valid. To state this result precisely it is necessary to extend the notion of fixed principal axes of strain. This idea makes no sense when two or more of the principal strains coalesce, for at those times the axes are not uniquely defined. The characteristic spaces, however, are uniquely defined, but change in type depending on the number $n(t)$ of distinct principal stretches: when $n(t) = 3$ the characteristic spaces are three mutually perpendicular lines; when $n(t) = 2$ the characteristic spaces are a line and a plane perpendicular to it; when $n(t) = 1$ the characteristic space is all of $\mathbf{R}^3$. Thus it only makes sense to demand that the characteristic spaces be independent of time on time intervals during which $n(t)$ remains constant.

Precisely then, let $n(t)$ denote the *number of distinct principal stretches* at time t. We say that the *right principal axes are essentially fixed* if the right stretch tensor $\mathbf{U}$ has characteristic spaces which are fixed in time on any time interval during which $n(t)$ is constant.

We are now in a position to state our main result.

Theorem 2. *The following are equivalent*:

(a) *The right principal axes are essentially fixed.*

(b) *The co-rotational derivative of* $\ln \mathbf{V}$ *corresponding to the left principal basis* $\{l_i\}$ *satisfies*

$$\mathbf{D} = (\ln \mathbf{V})^\circ.$$

(c) *The Jaumann derivative of* $\ln \mathbf{V}$ *satisfies*

$$\mathbf{D} = (\ln \mathbf{V})^*.$$

This theorem has the following obvious

Corollary. Suppose that the three principal stretches are distinct. Then the formulae

$$\mathbf{D} = (\ln \mathbf{V})^\circ, \qquad \mathbf{D} = (\ln \mathbf{V})^*$$

hold if and only if the three right principal axes are fixed for all time.

Of course, the right principal axes are the three lines generated by the basis vectors $\mathbf{r}_i$.

The next two lemmas facilitate the proof of Theorem 2. In these lemmas and in their proof, $\mathbf{\Omega}_r$ and $\mathbf{\Omega}_l$, respectively, denote the twirl tensors corresponding to the right and left principal bases, $\{\mathbf{r}_i\}$ and $\{\mathbf{l}_i\}$.

Lemma 1. *Let T be a time interval of nonzero length on which n(t) is constant. Then the following are equivalent*:

(i) *During T the characteristic spaces of* $\mathbf{U}$ *are independent of time.*

(ii) $\mathbf{\Omega}_r\mathbf{U} = \mathbf{U}\mathbf{\Omega}_r$ *on T.*

Proof. We begin by noting that

$$\begin{aligned}\mathbf{r}_i \cdot (\mathbf{U}\mathbf{\Omega}_r - \mathbf{\Omega}_r\mathbf{U})\mathbf{r}_j &= \mathbf{U}\mathbf{r}_i \cdot \mathbf{\Omega}_r\mathbf{r}_j - \mathbf{r}_i \cdot \mathbf{\Omega}_r\mathbf{U}\mathbf{r}_j \\ &= (\lambda_i - \lambda_j)\mathbf{r}_i \cdot \mathbf{\Omega}_r\mathbf{r}_j.\end{aligned} \tag{7}$$

Case 1 ($n = 3$). By (7), (ii) is equivalent to $\mathbf{\Omega}_r = \mathbf{0}$ on T, and since $\dot{\mathbf{r}}_i = \mathbf{\Omega}_r\mathbf{r}_i$, (i) and (ii) are equivalent.

Case 2 ($n = 2$). Here (using the spectral theorem) we may, without loss in generality, write $\mathbf{U}$ on T in the form

$$\mathbf{U}(t) = \lambda_1(t)\mathbf{r}_1(t) \otimes \mathbf{r}_1(t) + \lambda_2(t)\,[\mathbf{I} - \mathbf{r}_1(t) \otimes \mathbf{r}_1(t)],$$

so that the line spanned by $\mathbf{r}_1(t)$ and the plane perpendicular to $\mathbf{r}_1(t)$ are the characteristic spaces. Assume that (ii) holds. Then, since $\lambda_1 \neq \lambda_2$, we conclude from (7) that

$$\mathbf{r}_2 \cdot \mathbf{\Omega}_r\mathbf{r}_1 = \mathbf{r}_3 \cdot \mathbf{\Omega}_r\mathbf{r}_1 = 0 \tag{8}$$

on T. Thus

$$\dot{\mathbf{r}}_1 \cdot \mathbf{r}_2 = \dot{\mathbf{r}}_1 \cdot \mathbf{r}_3 = 0,$$

and since $\dot{\mathbf{r}}_1 \cdot \mathbf{r}_1 = 0$, we have $\mathbf{r}_1 = \text{constant}$ on T, which implies (i).

Conversely, if (i) holds, the above argument in reverse shows that (8) is valid. Condition (ii) then follows from (7), as $\lambda_2 = \lambda_3$ on T.

Case 3 ($n = 1$). Here

$$\mathbf{U}(t) = \lambda(t)\mathbf{I}$$

on T, and the conditions (i) and (ii) are satisfied identically.

This completes the proof of Lemma 1.

Lemma 2. *The following are equivalent*:

(i) *The right principal axes are essentially fixed.*

(ii) $\mathbf{\Omega}_r\mathbf{U} = \mathbf{U}\mathbf{\Omega}_r$ (*for all time*).

Proof. That (ii) implies (i) follows trivially from Lemma 1. If (i) holds, then $\mathbf{\Omega}_r$ commutes with $\mathbf{U}$ on all time intervals of nonzero length during which $n(t)$ is constant (by Lemma 1). In view of the continuity of the $\lambda_i(t)$, $n(t)$ is piecewise constant, and for any time t_0 at which $n(t)$ jumps there are right and left intervals (a, t_0) and (t_0, b) of nonzero length on which $n(t)$ is constant. Continuity of $\mathbf{\Omega}_r$ and $\mathbf{U}$ then gives (ii).

Proof of Theorem 2. $(a) \Leftrightarrow (b)$. By (4), $\mathbf{D} = (\ln \mathbf{V})^{\circ}$ is equivalent to $\text{sym}\,(\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1}) = 0$. Since

$$\mathbf{F}\mathbf{\Omega}_r\mathbf{F}^{-1} = \mathbf{R}(\mathbf{U}\mathbf{\Omega}_r\mathbf{U}^{-1})\mathbf{R}^T,$$

the latter condition is equivalent to the requirement that

$$U\mathbf{\Omega}_r\mathbf{U}^{-1} - \mathbf{U}^{-1}\mathbf{\Omega}_r\mathbf{U} = \mathbf{O},$$

or equivalently

$$\mathbf{U}^2\mathbf{\Omega}_r = \mathbf{\Omega}_r\mathbf{U}^2. \tag{9}$$

As is known (see, e.g. Ref. 2, p. 12) a symmetric tensor $\mathbf{A}$ commutes with a tensor $\mathbf{B}$ if and only if $\mathbf{B}$ leaves invariant the characteristic spaces of $\mathbf{A}$. Thus, since the characteristic spaces of $\mathbf{U}$ and $\mathbf{U}^2$ coincide, (9) is equivalent to (ii) and hence (i) of Lemma 2.

$(a) \Leftrightarrow (c)$. Assume $\mathbf{D} = (\ln \mathbf{V})^*$. Then, by (6),

$$(\ln \mathbf{V})(\mathbf{\Omega}_l - \mathbf{W}) - (\mathbf{\Omega}_l - \mathbf{W})(\ln \mathbf{V}) - \operatorname{sym}(\mathbf{F\Omega}_r\mathbf{F}^{-1}) = \mathbf{O}.$$

Since $\mathbf{W} = \mathbf{\Omega}_l - \operatorname{skw}(\mathbf{F\Omega}_r\mathbf{F}^{-1})$,

$$(\ln \mathbf{V})\operatorname{skw}(\mathbf{F\Omega}_r\mathbf{F}^{-1}) - \operatorname{skw}(\mathbf{F\Omega}_r\mathbf{F}^{-1})(\ln \mathbf{V}) - \operatorname{sym}(\mathbf{F\Omega}_r\mathbf{F}^{-1}) = \mathbf{O}.$$

Using $\mathbf{V} = \mathbf{RUR}^T$ and the isotropy of ln

$$\mathbf{R}[(\ln \mathbf{U})\operatorname{skw}(\mathbf{U\Omega}_r\mathbf{U}^{-1}) - \operatorname{skw}(\mathbf{U\Omega}_r\mathbf{U}^{-1})(\ln \mathbf{U}) - \operatorname{sym}(\mathbf{U\Omega}_r\mathbf{U}^{-1})]\mathbf{R}^T = \mathbf{O},$$

or equivalently,

$$(\ln \mathbf{U})(\mathbf{U\Omega}_r\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{\Omega}_r\mathbf{U}) - (\mathbf{U\Omega}_r\mathbf{U}^{-1} + \mathbf{U}^{-1}\mathbf{\Omega}_r\mathbf{U})(\ln \mathbf{U}) = \mathbf{U\Omega}_r\mathbf{U}^{-1} - \mathbf{U}^{-1}\mathbf{\Omega}_r\mathbf{U}.$$

Since $\mathbf{U}$ commutes with $\ln \mathbf{U}$,

$$(\ln \mathbf{U})(\mathbf{U}^2\mathbf{\Omega}_r + \mathbf{\Omega}_r\mathbf{U}^2) - (\mathbf{U}^2\mathbf{\Omega}_r + \mathbf{\Omega}_r\mathbf{U}^2)(\ln \mathbf{U}) = \mathbf{U}^2\mathbf{\Omega}_r - \mathbf{\Omega}_r\mathbf{U}^2. \tag{10}$$

As before, we will show that $\mathbf{U}$ commutes with $\mathbf{\Omega}_r$ by showing that $\mathbf{\Omega}_r$ leaves invariant the characteristic spaces of $\mathbf{U}$. Thus fix the time t, let λ denote a principal stretch and Λ the corresponding characteristic space for $\mathbf{U} = \mathbf{U}(t)$, choose $\mathbf{w} \in \Lambda$, and let

$$\mathbf{x} = \mathbf{\Omega}_r\mathbf{w}.$$

Applying (10) to $\mathbf{w}$ and noting that

$$\mathbf{U}^2\mathbf{w} = \lambda^2\mathbf{w}, \qquad (\ln \mathbf{U})\mathbf{w} = (\ln \lambda)\mathbf{w},$$

we arrive at

$$(\ln \mathbf{U} - (\ln \lambda)\mathbf{I})(\mathbf{U}^2\mathbf{x} + \lambda^2\mathbf{x}) = \mathbf{U}^2\mathbf{x} - \lambda^2\mathbf{x}. \tag{11}$$

Let β be a principal stretch with $\beta \neq \lambda$ and let $\mathbf{e}$ be a corresponding eigenvector, so that $\mathbf{e}\cdot\mathbf{w} = 0$. Then taking the inner product of (11) with $\mathbf{e}$ yields

$$(\ln \beta - \ln \lambda)(\beta^2 + \lambda^2)(\mathbf{e}\cdot\mathbf{x}) = (\beta^2 - \lambda^2)(\mathbf{e}\cdot\mathbf{x}).$$

Thus either $\mathbf{e}\cdot\mathbf{x} = 0$ or

$$\ln\frac{\beta}{\lambda} = \frac{\beta^2 - \lambda^2}{\beta^2 + \lambda^2}; \tag{12}$$

and, as we shall show at the end of the proof, (12) implies that $\beta = \lambda$. Thus $\mathbf{e}\cdot\mathbf{x} = 0$. We have shown that $\mathbf{x}$ is orthogonal to all characteristic spaces except Λ. Hence $\mathbf{x} \in \Lambda$ and $\mathbf{\Omega}_r$ leaves invariant the characteristic spaces of $\mathbf{U}$.

Conversely, if $\mathbf{\Omega}_r$ commutes with $\mathbf{U}$, then $\mathbf{\Omega}_r$ commutes with $\ln \mathbf{U}$. Since $\mathbf{U}$ commutes with $\ln \mathbf{U}$, eqn (10) holds trivially, and reversing the argument leading to (10) yields the condition $\mathbf{D} = (\ln \mathbf{V})^*$.

Thus (c) is equivalent to (ii) and hence (i) of Lemma 2.

To complete the proof we have only to show that (12) implies $\lambda = \beta$. Thus let $p = \ln(\beta/\lambda)$. Then (12) is equivalent the equation

$$p = \tanh p.$$

Since the derivative of tanh p is $\operatorname{sech}^2 p \le 1$, with equality holding only at $p = 0$, the graphs of p and tanh p intersect only at $p = 0$. This completes the proof of Theorem 2.

Remark. Some other relations which hold when the right principal axes are essentially fixed are

$$(\ln \mathbf{U})^{\cdot} = \dot{\mathbf{U}}\mathbf{U}^{-1} = \mathbf{U}^{-1}\dot{\mathbf{U}} = \mathbf{R}^T\mathbf{D}\mathbf{R}.$$

5. $(\ln \mathbf{V})^*$ APPROXIMATES $\mathbf{D}$

As we have noted in the last section, $\mathbf{D} = (\ln \mathbf{V})^*$ only in very special circumstances. We now show that $(\ln \mathbf{V})^*$ is, however, a very good approximation to $\mathbf{D}$ when the deformations are sufficiently small and slow. More precisely, consider a one-parameter family $\mathbf{F}_\epsilon(t)$ of deformation gradients, depending on a small parameter ϵ, and assume that the displacement gradient

$$\mathbf{H}_\epsilon = \mathbf{F}_\epsilon - \mathbf{I} \tag{13}$$

satisfies

$$\mathbf{H}_\epsilon = O(\epsilon), \qquad \dot{\mathbf{H}}_\epsilon = O(\epsilon) \tag{14}$$

as $\epsilon \to 0$. Here and in what follows we work at a particular time t, and for convenience we shall drop the subscript ϵ and the quantifier "as $\epsilon \to 0$" in subsequent equations.

Theorem 3. *The restrictions* (14) *imply that*

$$\mathbf{D} = (\ln \mathbf{V})^* + O(\epsilon^3). \tag{15}$$

Proof. We begin by listing three estimates which will be useful in what follows:

$$\begin{aligned}
(\mathbf{I}+\mathbf{A})^{-1} &= \mathbf{I} - \mathbf{A} + O(|\mathbf{A}|^2),\\
(\mathbf{I}+\mathbf{A})^{1/2} &= \mathbf{I} + \frac{1}{2}\mathbf{A} - \frac{1}{8}\mathbf{A}^2 + O(|\mathbf{A}|^3),\\
\ln(\mathbf{I}+\mathbf{A}) &= \mathbf{A} - \frac{1}{2}\mathbf{A}^2 + O(|\mathbf{A}|^3)
\end{aligned} \tag{16}$$

as $|\mathbf{A}| \to O$. In $(16)_1$, $\mathbf{A}$ is arbitrary; in $(16)_{2,3}$, $\mathbf{A}$ is symmetric.

Our next step will be to estimate the right side of the identity

$$(\ln \mathbf{V})^* = (\ln \mathbf{V})^{\cdot} + (\ln \mathbf{V})\mathbf{W} - \mathbf{W}\ln \mathbf{V}. \tag{17}$$

Let

$$\mathbf{E} = \operatorname{sym}\mathbf{H}, \qquad \mathbf{G} = \operatorname{skw}\mathbf{H}.$$

Then by (13),

$$\mathbf{V}^2 = \mathbf{F}\mathbf{F}^T = \mathbf{I} + 2\mathbf{E} + \mathbf{H}\mathbf{H}^T$$

and $(16)_{2,3}$ yield

$$\begin{aligned}
\mathbf{V} &= \mathbf{I} + \mathbf{E} + \frac{1}{2}(\mathbf{H}\mathbf{H}^T - \mathbf{E}^2) + O(\epsilon^3),\\
\ln \mathbf{V} &= \mathbf{E} + \frac{1}{2}\mathbf{H}\mathbf{H}^T - \mathbf{E}^2 + \mathbf{C}(\mathbf{H}), \qquad \mathbf{C}(\mathbf{H}) = O(\epsilon^3).
\end{aligned} \tag{18}$$

To derive an asymptotic expansion for $(\ln \mathbf{V})^{\cdot}$ we write $\mathbf{K}(\mathbf{H})$ for $\ln \mathbf{V}$ with $\mathbf{V}$ considered as a function of $\mathbf{H}$. Then $(18)_2$ is the Taylor expansion

$$\mathbf{K}(\mathbf{H}) = \mathbf{K}(\mathbf{O}) + \mathbf{K}'(\mathbf{O})[\mathbf{H}] + \frac{1}{2}\mathbf{K}''(\mathbf{O})[\mathbf{H}, \mathbf{H}] + \mathbf{C}(\mathbf{H}), \tag{19}$$

where $\mathbf{K}'(\mathbf{O})$ and $\mathbf{K}''(\mathbf{O})$ are the first and second (Frechet) derivatives of $\mathbf{K}$ at $\mathbf{O}$, with $\mathbf{K}'(\mathbf{O})[\mathbf{H}]$ linear in $\mathbf{H}$, $\mathbf{K}''(\mathbf{O})[\mathbf{H}_1, \mathbf{H}_2]$ symmetric and bilinear in $(\mathbf{H}_1, \mathbf{H}_2)$. If we differentiate (19) with respect to t, we arrive at

$$\mathbf{K}'(\mathbf{H})[\dot{\mathbf{H}}] = \mathbf{K}'(\mathbf{O})[\dot{\mathbf{H}}] + \mathbf{K}''(\mathbf{O})[\mathbf{H}, \dot{\mathbf{H}}] + \mathbf{C}(\mathbf{H})^{\cdot}.$$

On the other hand, if we expand $\mathbf{K}'(\mathbf{H})$ about $\mathbf{H} = \mathbf{O}$, using (14) we get

$$\mathbf{K}'(\mathbf{H})[\dot{\mathbf{H}}] = \mathbf{K}'(\mathbf{O})[\dot{\mathbf{H}}] + \mathbf{K}''(\mathbf{O})[\mathbf{H}, \dot{\mathbf{H}}] + O(\epsilon^3).$$

Hence

$$\mathbf{C}(\mathbf{H})^{\cdot} = O(\epsilon^3),$$

and differentiating $(18)_2$ with respect to time yields, after some work,

$$(\ln \mathbf{V})^{\cdot} = \dot{\mathbf{E}} - \operatorname{sym}(\dot{\mathbf{H}}\mathbf{H}) + \dot{\mathbf{G}}\mathbf{E} - \mathbf{E}\dot{\mathbf{G}} + O(\epsilon^3). \tag{20}$$

Next,

$$\begin{aligned} \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} &= \dot{\mathbf{H}}(\mathbf{I} - \mathbf{H} + O(\epsilon^2)) \\ &= \dot{\mathbf{H}} - \dot{\mathbf{H}}\mathbf{H} + O(\epsilon^3), \end{aligned}$$

and so

$$\begin{aligned} \mathbf{D} &= \operatorname{sym} \mathbf{L} = \dot{\mathbf{E}} - \operatorname{sym}(\dot{\mathbf{H}}\mathbf{H}) + O(\epsilon^3), \\ \mathbf{W} &= \operatorname{skw} \mathbf{L} = \dot{\mathbf{G}} + O(\epsilon^2). \end{aligned} \tag{21}$$

The estimates (17), $(18)_2$, (20) and (21) imply the desired result (15).

Remark. Since $\mathbf{D} = O(\epsilon)$ and $(\ln \mathbf{V})^* = O(\epsilon)$, the asymptotic expansion $\mathbf{D} = (\ln \mathbf{V})^* + O(\epsilon^3)$ shows $(\ln \mathbf{V})^*$ to be an excellent approximation to $\mathbf{D}$ when $\mathbf{H}$ and $\dot{\mathbf{H}}$ are small.

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